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ANTIPODAL VECTOR BUNDLE MONOMORPHISMS

BY

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ABSTRACT

In this paper we study existence and classification questions concerning antipodal vector bundle monomorphisms u (i.e., u is regularly homotopic to its negative $-u$). In a metastable dimension range the singularity approach yields complete obstructions which, however, have to be weakened usually in order to become computable. In many situations we determine the resulting "weak, stable" invariants completely; a central role here is played by the antipodality obstruction $v_i(\alpha, \beta)$, a curious combination of Stiefel–Whitney classes. Moreover, in some sample cases we describe precisely how much information gets lost by the transition to these weaker invariants. This involves, e.g., identifying some classical second order obstructions. As an application we exhibit a setting where the difference invariant $d(u, -u)$ distinguishes all (and in fact, infinitely many) regular homotopy classes. Also, we give complete existence and enumeration results for nonstable and stable tangent plane fields on complex projective spaces in terms of explicit numerical conditions.

I. Introduction and statement of results

Let α^a and β^b denote real vector bundles of the indicated dimensions $a \leq b$ over a connected closed smooth n-dimensional manifold N.

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Definition: A vector bundle monomorphism $u\colon \alpha \hookrightarrow \beta$ (i.e., an injective continuous vector bundle homomorphism over the identity map id_N) is called **antipodal** if u and $-u$ are regularly homotopic (i.e., homotopic through vector bundle monomorphisms).

In this paper we are concerned at first with the existence problem of such antipodal monomorphisms. This amounts to the following two partial questions which we will attack by the singularity method (cf. [K 1]).

Question A: Is there *any* monomorphism $u\colon \alpha \hookrightarrow \beta$?

If yes, it restricts to an injective linear map on every line in any of the fibers $\alpha_x, x \in N$. In other words, if u exists then there is also a monomorphism from the canonical line bundle λ over the projectification $P(\alpha)$ of α into the pullback of β; equivalently, the vector bundle $\mathrm{Hom}(\lambda, \beta) \cong \lambda \otimes \beta$ over $P(\alpha)$ allows a nowhere vanishing section s_u.

Now, given *any* generic section s of $\lambda \otimes \beta$ we obtain the following *singularity data:*

(i) the $(n + a - 1 - b)$-dimensional manifold $Z := s^{-1}\{0\}$ formed by the zero set of s;

(ii) the continuous map $g\colon Z \subset P(\alpha)$; and

(iii) the stable vector bundle isomorphism

$$\overline{g} : TZ \oplus g^*(\lambda \otimes \beta \oplus \mathbb{R}) \underset{\sim}{\cong} g^*(\lambda \otimes \alpha \oplus TN) \tag{I.1}$$

(which is deduced from the obvious identification of $\lambda \otimes \beta \mid Z$ with the normal bundle of Z in $P(\alpha)$). The resulting *normal bordism* class

$$\omega(\alpha, \beta) = [Z, g, \overline{g}] \in \Omega_{n+a-b-1}(P(\alpha); \phi) \tag{I.2}$$

(with coefficients in the virtual vector bundle

$$\phi := \lambda \otimes \beta - \lambda \otimes \alpha - TN \tag{I.3}$$

over $P(\alpha)$) is an obstruction to the existence of nowhere vanishing sections in $\lambda \otimes \beta$ and hence of monomorphisms from α to β. Actually, in a certain "metastable" dimension range it fully answers our first question.

THEOREM A (cf. [K 1], 2.15 and 3.7): *Assume $n < 2(b - a)$. Then a monomorphism $u\colon \alpha \hookrightarrow \beta$ exists if and only if $\omega(\alpha, \beta) = 0$.*

Question B: Given a monomorphism $u\colon \alpha \hookrightarrow \beta$, when is it regularly homotopic to its negative $-u$?

In this situation the singularity approach outlined above yields the *difference obstruction*

$$d(u,-u) \in \Omega_{n+a-b}(P(\alpha);\phi) \tag{I.4}$$

which again settles our question in a suitable dimension range.

THEOREM B (cf. [K 1], 4.14): *Assume $n+1 < 2(b-a)$. Then a monomorphism $u\colon \alpha \hookrightarrow \beta$ is antipodal if and only if $d(u,-u)=0$.*

Actually, in these dimensions monomorphisms u — if they exist — are completely classified by the difference obstruction $d(u_0,u)$ where u_0 is fixed. But this is an *external* invariant based on the comparison with some other, arbitrarily selected monomorphism u_0. It would be desirable to distinguish monomorphisms u by their *internal* geometry, e.g., by the properties of the complement of $u(\alpha)$ in β or by the antipodality obstruction. In particular, we will be interested in the question how many different regular homotopy classes the internal invariant $d(u,-u)$ can detect.

We can bring together most of the existence and classification aspects of the obstructions discussed above into one unifying setting. Let ξ denote the nontrivial line bundle over the circle S^1.

Question C: When is there a monomorphism

$$\widetilde{u} : \widetilde{\alpha} := \xi \otimes \alpha \hookrightarrow \beta$$

over $\widetilde{N} := S^1 \times N$? (We drop obvious pullbacks from the notation.)

Note that the a-plane bundle $\widetilde{\alpha}$ over $S^1 \times N$ can be interpreted as the mapping torus of the antipodal isomorphism $-\operatorname{id}_\alpha$ over id_N. Thus the required monomorphism $\widetilde{u}$ consists basically of a monomorphism $u\colon \alpha \hookrightarrow \beta$ over N, together with a regular homotopy from u to $u \circ (-\operatorname{id}_\alpha) = -u$. As before the singularity method yields the obstruction

$$\omega(\widetilde{\alpha},\beta) \in \Omega_{n+a-b}(P(\widetilde{\alpha});\widetilde{\phi}) \tag{I.5}$$

in the normal bordism of $P(\widetilde{\alpha})$ with coefficients in the virtual vector bundle

$$\widetilde{\phi} := \widetilde{\lambda} \otimes \beta - \widetilde{\lambda} \otimes \widetilde{\alpha} - TN \tag{I.6}$$

(where $\widetilde{\lambda}$ denotes the canonical line bundle over $P(\widetilde{\alpha})$).

THEOREM C: *An antipodal monomorphism* $u : \alpha \hookrightarrow \beta$ *over* N *exists if and only if there is (any) monomorphism* $\widetilde{u}\colon \widetilde{\alpha} \hookrightarrow \beta$ *over* $S^1 \times N$. *In the dimension range* $n+1 < 2(b-a)$ *this holds if and only if* $\omega(\widetilde{\alpha}, \beta) = 0$.

It is the purpose of this paper to investigate and exploit the three obstructions $\omega(\alpha, \beta), d(u, -u)$ and $\omega(\widetilde{\alpha}, \beta)$ (occurring in Theorems A, B, and C) and their relations among each other and with classical invariants. A central tool is the following commuting diagram (which will be established and discussed in sections 1 and 4; see also (I.12) and (I.13) below).

$$\begin{array}{lcccl}
d(u_0,u) \in & \Omega_k(P(\alpha);\phi) & \xrightarrow{\text{forg}_k} & \overline{\Omega}_k(P^\infty \times N;\phi) & \\
\downarrow & \downarrow \partial_* & & \downarrow \overline{\partial}_* & \\
d(u,-u) \in & \Omega_k(P(\alpha);\phi) & \xrightarrow{\text{forg}_k} & \overline{d}(u,-u) \in \overline{\Omega}_k(P^\infty \times N;\phi) & \cong \begin{cases} \mathfrak{N}_k(N) & \text{if } a \not\equiv b \pmod 2; \\ \overline{\Omega}_k(N;\eta) \oplus \mathfrak{N}_{k-1}(N) & \text{else} \end{cases} \\
\downarrow & \downarrow \text{incl}_* & & \downarrow \overline{\text{incl}}_* & \\
\omega(\widetilde{\alpha},\beta) \in & \Omega_k(P(\widetilde{\alpha});\widetilde{\phi}) & \xrightarrow{\widetilde{\text{forg}}_k} & \overline{\omega}(\widetilde{\alpha},\beta) \in \overline{\Omega}_k(P^\infty \times \widetilde{N};\widetilde{\phi}) & \cong \begin{cases} \mathfrak{N}_k(\widetilde{N}) & \text{if } a \not\equiv b \pmod 2; \\ \overline{\Omega}_k(\widetilde{N};\widetilde{\eta}) \oplus \mathfrak{N}_{k-1}(\widetilde{N}) & \text{else} \end{cases} \\
\downarrow & \downarrow \pitchfork & & \downarrow \overline{\pitchfork} & \\
\omega(\alpha,\beta) \in & \Omega_{k-1}(P(\alpha);\phi) & \xrightarrow{\text{forg}_{k-1}} & \overline{\omega}(\alpha,\beta) \in \overline{\Omega}_{k-1}(P^\infty \times N;\phi) & \cong \begin{cases} \mathfrak{N}_{k-1}(N) & \text{if } a \not\equiv b \pmod 2; \\ \overline{\Omega}_{k-1}(N;\eta) \oplus \mathfrak{N}_{k-2}(N) & \text{else} \end{cases} \\
 & \downarrow & & \downarrow & \\
 & \vdots & & \vdots &
\end{array} \qquad \text{(I.7)}$$

DIAGRAM (I.7)

Here $\widetilde{N} := S^1 \times N$. Both vertical long exact sequences involve Gysin isomorphisms (cf. §1). For $k = n + a - b$ our obstructions fit in as indicated:

$$\pitchfork (\omega(\widetilde{\alpha}, \beta)) = \omega(\alpha, \beta); \tag{I.8}$$

moreover, if a monomorphism $u\colon \alpha \hookrightarrow \beta$ exists and hence $\omega(\alpha, \beta) = 0$, then

$$\text{incl}_*(d(u, -u)) = \omega(\widetilde{\alpha}, \beta); \tag{I.9}$$

finally, for any two monomorphisms $u_0, u\colon \alpha \hookrightarrow \beta$ we have

$$\pm\partial_*(d(u_0, u)) = d(u, -u) - d(u_0, -u_0) \tag{I.10}$$

($= d(u, -u)$ if u_0 happens to be antipodal).

Thus diagram (I.7) is also very relevant for the classification problem.

Question D: How many different antipodal monomorphisms $u\colon \alpha \hookrightarrow \beta$ exist? What is their percentage among *all* monomorphisms?

Clearly, if $(u_t\colon \alpha \hookrightarrow \beta)_{t\in I}$ is any regular homotopy of monomorphisms and if u_0 is antipodal, then so is u_t for every $t \in I$. In particular, classifying antipodal monomorphisms up to standard regular homotopy amounts to classifying them up to deformations through antipodal monomorphisms.

THEOREM D: (i) *If* $n < 2(b-a)$ *then*

$$\{d(u,-u) \in \Omega_{n+a-b}(P(\alpha),\phi) \mid u\colon \alpha \hookrightarrow \beta\} = \mathrm{incl}_*^{-1}(\omega(\widetilde{\alpha},\beta));$$

if even $n+1 < 2(b-a)$ *and if* ∂_{n+a-b} *is injective, then the antipodality obstruction* $d(u,-u)$ *classifies all monomorphisms* $u\colon \alpha \hookrightarrow \beta$ *completely up to regular homotopy.*

(ii) *Assume that* $n+1 < 2(b-a)$ *and that an antipodal monomorphism* u_0 *exists. Then the difference invariant* $d(u_0,-)$ *establishes a bijective correspondence between all regular homotopy classes of antipodal monomorphisms and the elements of* $\ker \partial_{n+a-b}$.

In particular, all monomorphisms from α *to* β *are antipodal if and only if* $\partial_{n+a-b} \equiv 0$ *(or, equivalently,* incl_{n+a-b} *is injective).*

In general, the smaller $\ker(\mathrm{incl}_*) = \partial_*(\Omega_{n+a-b}(P(\alpha);\phi))$ the bigger the relative size of $\ker \partial_{n+a-b}$ and hence the percentage of antipodal among all monomorphisms. The two extreme opposites spelled out in Theorem D can actually occur. The case $\partial_* \equiv 0$ is discussed in detail in [KMS]; actually, in view of the identity (1.3) below, it can also be described by some kind of "antipodality condition on the bordism level". On the other hand, in Examples G and H and in section 5 below we will encounter various concrete situations where ∂_* has a very small kernel; in particular, we will exhibit a setting where $d(u,-u)$ is a complete *internal* invariant which distinguishes all (and, in fact, infinitely many) regular homotopy classes of monomorphisms.

Finiteness questions arising in this context are settled quite generally by the following result which will be proved in §2.

COROLLARY: (i) *Assume that* $\omega(\alpha,\beta) = 0$ *and* $n < 2(b-a)$. *Then the invariant* $d(u,-u)$ *distinguishes infinitely many regular homotopy classes of monomorphisms* $u\colon \alpha \hookrightarrow \beta$ *if and only if* a *and* b *are odd and* $H_{n+a-b}(N;\widetilde{\mathbb{Q}}_\eta) \neq 0$.

(ii) *Assume that* $\omega(\widetilde{\alpha},\beta) = 0$ *and* $n+1 < 2(b-a)$. *Then there are infinitely many regular homotopy classes of* antipodal *monomorphisms if and only if* b *is even and one of the following two conditions hold:*

1) *a is odd and* $H_{n-b+1}(N;\widetilde{\mathbb{Q}}_{\beta-TN}) \neq 0$*; or:*
2) *a is even and* $\ker \tau^{\mathbb{Q}}_{n+a-b} \oplus \operatorname{coker} \tau^{\mathbb{Q}}_{n+a-b+1} \neq 0$
(where—up to isomorphisms—

$$\tau^{\mathbb{Q}}_i \colon H_i(N;\widetilde{\mathbb{Q}}_\eta) \longrightarrow H_{i-a}(N;\widetilde{\mathbb{Q}}_{\beta-TN}) \tag{I.11}$$

denotes the homomorphism which occurs in the Gysin sequence of α*).*

Here and later, given a virtual vector bundle γ over a suitable space X, the twisted rational (or integer, resp.) coefficient system corresponding to the orientation line bundle ξ_γ of γ (i.e., $w_1(\xi_\gamma) = w_1(\gamma)$) is denoted by $\widetilde{\mathbb{Q}}_\gamma$ (or $\widetilde{\mathbb{Z}}_\gamma$, resp.). Moreover, we put for short

$$\eta := \beta - \alpha - TN \in KO(N) \quad \text{and} \quad \widetilde{\eta} := \beta - \widetilde{\alpha} - TN \in KO(\widetilde{N}). \tag{I.12}$$

Our singularity obstructions and the normal bordism groups in which they lie are strong but usually hard to compute. Therefore it is meaningful to also study weakened versions which are more accessible. For example, we may replace stable vector bundle isomorphisms such as $\overline{g}$ in (I.1) by the corresponding isomorphisms $\xi_Z \cong g^*(\xi_\phi)$ of orientation line bundles; similarly, we may forget that the canonical line bundle λ over $P(\alpha)$ lies in the pullback of α and retain it only as an abstract line bundle with classifying map into infinite projective space P^∞. This procedure defines the forgetful homomorphisms forg_* and $\widetilde{\text{forg}}_*$ in the diagram (I.7) above as well as their target groups. For example, if ϕ is orientable, we are just dealing with the usual oriented bordism groups of $P^\infty \times N$. But in any case the resulting "twisted" oriented bordism groups can be entirely described —as indicated in (I.7)— by unoriented and twisted oriented bordism groups of N and $\widetilde{N}$ (see [K 3], 1.3).

Often this allows us to evaluate the "weak stable versions"

$$\begin{aligned} \overline{d}(u,-u) &:= \text{forg}_{n+a-b}(d(u,-u)), \\ \overline{\omega}(\widetilde{\alpha},\beta) &:= \widetilde{\text{forg}}_{n+a-b}(\omega(\widetilde{\alpha},\beta)), \quad \text{and} \\ \overline{\omega}(\alpha,\beta) &:= \text{forg}_{n+a-b-1}(\omega(\alpha,\beta)) \end{aligned} \tag{I.13}$$

of our three obstructions (compare (I.7)).

For this purpose we introduce the following combination of (dual) Stiefel–Whitney classes of β (and α):

$$v_i(\alpha,\beta) := \sum_{j=0}^{i} (a+j)\overline{w}_j(\alpha) w_{i-j}(\beta) \in H^i(N;\mathbb{Z}_2), \quad i \in \mathbb{Z}. \tag{I.14}$$

(Since this sum involves only every other summand of $w_i(\beta - \alpha) = \Sigma\overline{w}_j(\alpha)w_{i-j}(\beta)$, we may, in a way, interpret $v_i(\alpha,\beta)$ as being "one half of $w_i(\beta - \alpha)$". The relevance of both classes stems from the fact that they occur as components of $w(\beta - \widetilde{\alpha}) \in H^*(S^1 \times N; \mathbb{Z}_2)$, cf. formula 3.5 below.)

If $a \equiv b(2)$ (and $a < b$), consider also the classical primary obstructions (in the sense of [S], 35.3)

$$\text{(I.15)} \quad \overline{c}(\alpha,\beta) \in H^{b-a+1}(N; \widetilde{\mathbb{Z}}_{\beta-\alpha}) \quad \text{and} \quad \overline{c}(\widetilde{\alpha},\beta) \in H^{b-a+1}(S^1 \times N; \widetilde{\mathbb{Z}}_{\beta-\widetilde{\alpha}})$$

to sectioning the obvious bundles $\cup\,\mathrm{Mono}(\alpha_x, \beta_x)$ and $\cup\,\mathrm{Mono}_{\widetilde{x}}(\widetilde{\alpha}_{\widetilde{x}}, \beta_{\widetilde{x}})$, i.e., to finding monomorphisms u: $\alpha \hookrightarrow \beta$ and $\widetilde{u}$: $\widetilde{\alpha} \hookrightarrow \beta$, respectively. Similarly, if u exists, let

$$\text{(I.16)} \qquad \overline{c}(u, -u) \in H^{b-a+1}(N \times (I, \partial I); \widetilde{\mathbb{Z}}_{\beta-\alpha})$$

be the primary obstruction to deforming u through monomorphisms into $-u$.

The following table lists necessary conditions (to the right) for the vanishing of the weak stable versions of our three obstructions (as indicated to the left hand side).

<table>
<tr><td>TABLE I.17</td><td>if $a \not\equiv b(2)$</td><td colspan="2">if $a \equiv b(2)$ (and $a < b$)</td></tr>
<tr><td>$\overline{d}(u, -u) = 0$</td><td>$v_i(\alpha,\beta) = 0$
for $i \geq b - a$</td><td colspan="2">$v_i(\alpha,\beta) = 0$ for $i > b - a$ and
$\overline{c}(u, -u) = 0$</td></tr>
<tr><td rowspan="2">$\overline{\omega}(\widetilde{\alpha},\beta) = 0$</td><td rowspan="2">$v_{i-1}(\alpha,\beta) = 0$ and
$w_i(\beta - \alpha) = 0$ for $i > b - a$</td><td colspan="2">$v_{i-1}(\alpha,\beta) = 0 = w_i(\beta - \alpha)$ for
$i > b - a + 1$ and $\overline{c}(\alpha,\beta) = 0$ and</td></tr>
<tr><td>$v_{b-a}(\alpha,\beta) = 0$
if $a \equiv b \not\equiv 0(2)$</td><td>$\overline{c}(\widetilde{\alpha},\beta) = 0$
if $a \equiv b \equiv 0(2)$</td></tr>
<tr><td>$\overline{\omega}(\alpha,\beta) = 0$</td><td>$w_i(\beta - \alpha) = 0$
for $i > b - a$</td><td colspan="2">$w_i(\beta - \alpha) = 0$ for $i > b - a + 1$
and $\overline{c}(\alpha,\beta) = 0$</td></tr>
</table>

THEOREM E: *We have* $2 \cdot \overline{\omega}(\alpha,\beta) = 0$, $2 \cdot \overline{\omega}(\widetilde{\alpha},\beta) = 0$ *(and* $2 \cdot \overline{d}(u, -u) = 0$ *in case* a *or* b *is even).*

Furthermore, if

(i) $a \not\equiv b(2)$; *or*

(ii) $n + a - b \leq 4$; *or*

(iii) $w_1(N) + w_1(\alpha) + w_1(\beta) = 0$ *and the torsion of* $H_*(N;\mathbb{Z})$ *consists of elements of order* 2;

then the necessary conditions listed in the table above are also sufficient.

This and the following result will be proved in section 4 below.

PROPOSITION F: *If a or b is even, then the homomorphism* $\overline{\text{incl}_*}$ (cf. I.7) *is split injective. Hence* $\bar{d}(u,-u)$ *is independent of* u: $\alpha \hookrightarrow \beta$ *(in fact,* $\overline{\text{incl}_*}(\bar{d}(u,-u))$ *coincides with a component of* $\overline{\omega}(\widetilde{\alpha},\beta)$ *which is well-defined even if no monomorphism* u *exists).*

As this whole discussion shows, the forgetful homomorphisms forg_k and $\widetilde{\text{forg}}_k$ in diagram (I.7) describe essentially the transition from our sharp obstructions $\omega(\alpha,\beta)$, $\omega(\widetilde{\alpha},\beta)$ and $d(u,-u)$ to classical (first order) obstructions. Thus the more subtle aspects (often of higher order) are mainly concentrated in the kernels of forg_k and $\widetilde{\text{forg}}_k$. Fortunately, we have some control over these kernels. For example, they are always finite if b is odd or if $k+1<a$ (for precise criteria see Proposition 2.10 below). More importantly, for low $k=n-b+a\leq a$ our forgetful homomorphisms often fit into exact sequences which allow explicit calculations (see [K 3], theorem 3.1, and especially [K 1], theorem 9.3 and the "toolkit" assembled there).

As an illustration of the potential power of all these techniques we discuss the case $a=3, b=n+1$ in some detail in §5. Exploiting the deep interplay of existence and classification aspects which pervades the whole theory and has a focus, e.g., in the first order obstructions $v_i(\alpha,\beta)$ we can compute also two subtle second order obstructions in the proof of Theorem 5.10. This leads to a precise vanishing criterion for $\omega(\widetilde{\alpha},\beta)$ and allows us in interesting cases to deduce complete existence and enumeration results in terms of explicit numerical conditions.

Example G: Given integers $q>2$, p and $p_1,\ldots,p_q$, consider the vector bundles

$$\alpha^3=\lambda_p\oplus\underset{\sim}{\mathbb{R}} \quad \text{and} \quad \beta^{n+1}=\lambda_{p_1}\oplus\ldots\oplus\lambda_{p_q}\oplus\underset{\sim}{\mathbb{R}}$$

over complex projective space $N^n=\mathbb{C}P(q)$ (where λ_k denotes the k-fold complex tensor power of the canonical complex line bundle).

Then there exists a monomorphism u: $\alpha\hookrightarrow\beta$ (antipodal or not) precisely if $\prod(p_i-p)$ is divisible by 4 in the case $p+\Sigma p_i\not\equiv q\equiv 0(2)$ and $\prod(p_i-p)$ is even otherwise. If this is satisfied the invariant $d(u,-u)$ distinguishes infinitely many monomorphisms in a $1:1$ (or $2:1$) fashion according as $\Sigma p_i\equiv(p+1)q$ (2) (or not, resp.).

An *antipodal* monomorphism exists precisely if p_i-p and p_j are even for some $1\leq i\neq j\leq q$ and—in addition in the case $p+\Sigma p_i\equiv q$ (2) —the Euler class

$e(u(\alpha)^{\perp}) \in H^{n-2}(N;\mathbb{Z})$ of the cokernel bundle of any monomorphism $u\colon \alpha \hookrightarrow \beta$ is divisible by 4. ∎

This is shown at the end of §5. As another illustration we establish (in §6) complete numerical criteria for nonstable and stable tangent plane fields on $\mathbb{C}P(q)$.

Example H: Given $q \geq 2$ and $p \in \mathbb{Z}$, let λ_p again denote the p-fold tensor power of the canonical complex line bundle over complex projective space $N = \mathbb{C}P(q)$. Then:

(i) There exists a *complex* vector bundle monomorphism from λ_p to the tangent bundle $T\mathbb{C}P(q)$ if and only if $p = -2$ and q is odd (and then the number of complex regular homotopy classes is 2).

(ii) There exists a *real* vector bundle monomorphism from λ_p to $T\mathbb{C}P(q)$ if and only if p is even, q is odd and $q+1$ is an even (or odd, resp.) multiple of p according as $q \equiv 3(4)$ (or $q \equiv 1(4)$, resp.), and then the number of real regular homotopy classes is 4.

(iii) There exists a monomorphism u from $\lambda_p \oplus \underset{\sim}{\mathbb{R}}$ to $T\mathbb{C}P(q) \oplus \underset{\sim}{\mathbb{R}}$ if and only if p is even and q is odd. When this holds, the antipodality obstruction $d(u,-u)$ distinguishes infinitely many different such monomorphisms u, and precisely two regular homotopy classes $[u], [u']$ can have the same value $d(u,-u) = d(u',-u')$.

(iv) There exists an *antipodal* monomorphism from $\lambda_p \oplus \underset{\sim}{\mathbb{R}}$ to $T\mathbb{C}P(q) \oplus \mathbb{R}$ if and only if p is even and $q \equiv 3(4)$, and then their number is 2 (up to regular homotopy).

This yields many concrete situations where real monomorphisms but no antipodal (or complex) monomorphisms exist. Also, observe the effect of stabilization: there are infinitely many *nonantipodal* monomorphisms in (iii) even though *all* monomorphisms in (i) and (ii) are antipodal due to the complex structures.

Remark I.18: The existence criteria in Example H still hold in the case $q = 1$. However, the resulting vector bundle isomorphisms over $\mathbb{C}P(1) = S^2$ are uniquely determined up to isotopy (and — in (iii) — the obvious reflection).

The results and techniques of this paper can also be applied to obtain a complete homotopy classification of line fields or, equivalently, of Lorentz metrics, e.g., on all closed manifolds of dimension $n \equiv 0(4)$. Details will be given in [K 5].

NOTATIONS AND CONVENTIONS. All (co-)homology has coefficients in $\mathbb{Z}_2$ unless specified differently; e.g., $x = w_1(\lambda) \in H^1(P^k), k \geq 1$, and $y \in H^1(S^1)$ denote the generators where λ is the canonical line bundle over real projective space P^k. Over any base space $\underset{\sim}{\mathbb{R}}^k$ denotes the trivial k-plane bundle. For any vector bundle

γ the multiplicative inverse of its (total) Stiefel–Whitney class $w(\gamma)$ is denoted by $\overline{w}(\gamma)$; e.g., for any virtual vector bundle $\gamma - \delta$ we have $w(\gamma-\delta) = w(\gamma)\overline{w}(\delta)$. Pullbacks (e.g., of vector bundles or cohomology classes) are often not written. In order to simplify our notation we omit also the superscript *st* which was used in [K 3] (e.g., by defining $\overline{\omega}^{st}(\alpha,\beta) = \mathrm{forg}(\omega(\alpha,\beta)))$ to indicate that forg has also a stabilizing aspect.

§1. The main diagram and the strong invariants

In this section we establish basic facts about the diagram (I.7) and use it to clarify the relations between our ("strong") normal bordism invariants. In particular, we prove formulas (I.8), (I.9), (I.10) and Theorem D.

As in Question C consider the nontrivial line bundle ξ over S^1 and the a-plane bundle $\widetilde{\alpha} = \xi \otimes \alpha$ over $\widetilde{N} = S^1 \times N$ together with its projectification $P(\widetilde{\alpha})$ (i.e., the manifold consisting of all lines through 0 in any of the fibers $\widetilde{\alpha}_x, x \in \widetilde{N}$). Since $\widetilde{\alpha}, P(\widetilde{\alpha})$ and $\widetilde{\lambda}$ are the mapping tori of $-\mathrm{id}_\alpha, \mathrm{id}_{P(\alpha)}$ and $-\mathrm{id}_\lambda$, resp., we have the natural identifications

$$
\begin{array}{ccc}
\tilde{\lambda} & = & \xi \otimes \lambda \\
\downarrow & & \downarrow \\
P(\tilde{\alpha}) & = & S^1 \times P(\alpha)
\end{array}
\tag{1.1}
$$

(λ and $\widetilde{\lambda}$ denote the canonical line bundles over $P(\alpha)$ and $P(\widetilde{\alpha})$, resp.; here and in the remainder of this paper we drop obvious pullbacks from the notation). Thus we conclude

$$
\widetilde{\phi} = \xi \otimes \lambda \otimes \beta - \lambda \otimes \alpha - TN \tag{1.2}
$$

(compare (I.6)); intuitively speaking, $\widetilde{\phi}$ is "twisted around S^1" via the antipodal map $-\mathrm{id}_\beta$ on β.

Now consider the exact normal bordism sequence of the pair $(S^1, S^1-*)\times P(\alpha)$ and use the "Gysin" isomorphism

$$
\pitchfork : \Omega_{*+1}(S^1 \times P(\alpha), (S^1 - *) \times P(\alpha); \widetilde{\phi}) \xrightarrow{\cong} \Omega_*(P(\alpha); \phi)
$$

defined by taking transverse intersections with the submanifold $\{*\} \times P(\alpha)$ (cf. [D], chapitre I, 3.1, 3.3 and §6). We obtain the vertical exact sequence on the left hand side in diagram (I.7) (and, by analogy, also the one to the right hand side). Inspecting the inverses of the isomorphisms $\pitchfork$ and i_* (where $i : P(\alpha) \subset$

$(S^1 - *) \times P(\alpha)$ is an obvious inclusion) we see that the boundary endomorphism takes the form

$$\partial_* = \pm((-\operatorname{id}_\beta)_* - \operatorname{id}) \tag{1.3}$$

on $\Omega_*(P(\alpha);\phi)$ (compare (1.2)); here $(-\operatorname{id}_\beta)_*$ composes vector bundle isomorphisms $\overline{g}$ (as in (I.1)) with the antipodal map on β.

When we intersect $P(\alpha) = \{*\} \times P(\alpha)$ in $S^1 \times P(\alpha)$ with the zero set of a suitable section $\widetilde{s}$ in $\widetilde{\lambda} \otimes \beta$ we get the zero set of the restricted section $\widetilde{s} \mid P(\alpha)$ in $\lambda \otimes \beta$. This implies formula (I.8) in the introduction.

Similarly, given any monomorphism $u\colon \alpha \hookrightarrow \beta$ over N and its induced section s_u in the vector bundle $\lambda \otimes \beta$ over $P(\alpha)$, a (singular) homotopy from s_u to $-s_u$ has the same zero set as the resulting section in the mapping torus of $-\operatorname{id}_{\lambda\otimes\beta}$. Formula (I.9) follows easily.

Moreover, for any monomorphisms $u_0, u\colon \alpha \hookrightarrow \beta$ we have

$$d(-u_0, -u) = (-\operatorname{id}_\beta)_*\,(d(u_0, u))$$

and therefore (by (1.3))

$$\begin{aligned} d(u,-u) &= d(u,u_0) + d(u_0,-u_0) + d(-u_0,-u) \\ &= d(u_0,-u_0) \pm \partial_*(d(u_0,u)). \end{aligned} \tag{1.4}$$

This establishes formula (I.10).

Next we prove Theorem D of the introduction. If $n < 2(b-a)$ and $c \in \operatorname{incl}_*^{-1}(\omega(\widetilde{\alpha},\beta))$, then there exists a monomorphism $u_0\colon \alpha \hookrightarrow \beta$ and every element of $\Omega_{n+a-b}(P(\alpha);\phi)$ can still be realized as the difference invariant $d(u_0, u)$ for some further monomorphism u; this follows as in the proof of theorem 4.8 in [K 1]. In particular, the element

$$c - d(u_0, -u_0) \in \ker(\operatorname{incl}_*) = \partial_*(\Omega_{n+a-b}(P(\alpha);\phi))$$

can be written as $\pm\partial_*(d(u_0,u))$. Thus, by (I.10),

$$c = d(u_0,-u_0) \pm \partial_*(d(u_0,u)) = d(u,-u)$$

as claimed.

Now assume even that $n+1 < 2(b+a)$. Then the difference invariant $d(u_0,u)$ classifies monomorphisms u (cf. [K 1], 4.14) and so does $d(u,-u)$ by (I.10) if ∂_* is injective. If u_0 is antipodal, all other antipodal monomorphisms are characterized by

$$d(u,-u) = \pm\partial_*(d(u_0,u)) = 0$$

as claimed. ∎

Finally, we give another interpretation of the antipodality obstruction $d(u,-u)$. Any monomorphism $u\colon \alpha \hookrightarrow \beta$ over N induces a monomorphism from the canonical line bundle λ over $P(\alpha)$ into (the pullback of) β and hence a nowhere vanishing section s_u of $\lambda\otimes\beta$. We obtain a decomposition $\lambda\otimes\beta = \underset{\sim}{\mathbb{R}}\oplus(\mathbb{R}s_u)^\perp$ into the span of s_u and its $(b-1)$-dimensional complement over $P(\alpha)$. Consider the obstruction

$$\omega(\underset{\sim}{\mathbb{R}}, (\mathbb{R}s_u)^\perp) \in \Omega_{n+a-b}\left(P(\alpha); (\mathbb{R}s_u)^\perp - TP(\alpha) = \phi\right)$$

to a section without zeroes in this complement (compare (I.2) and (I.3)).

PROPOSITION 1.5: *For any monomorphism $u\colon \alpha\hookrightarrow\beta$ over N we have*

$$d(u,-u) = \pm\omega(\underset{\sim}{\mathbb{R}}, (\mathbb{R}s_u)^\perp).$$

Proof: Given a generic section s of $(\mathbb{R}s_u)^\perp$ over $P(\alpha)$, define a section S of $\lambda\otimes\beta$ over $P(\alpha)\times I$ by

$$S(x,t) = \cos(\pi t)s_u(x) + \sin(\pi t)s(x).$$

This is a generic homotopy from s_u to $-s_u = s_{-u}$. Its singularity data (at $t=\frac{1}{2}$) coincide with those of s. ∎

§2. Rational normal bordism theory

In this section we prove the corollary to Theorem D of the introduction as well as finiteness criteria for the kernel of forg_k (cf. (I.7)). We achieve this by tensorizing the relevant normal bordism groups with the field $\mathbb{Q}$ of rational numbers.

Let X be a CW-complex all of whose skeletons are compact and let $\psi = \psi_+ - \psi_-$ be a virtual vector bundle over X. Recall that the rational Hurewicz homomorphism

$$\mu_i^{\mathbb{Q}}\colon \Omega_i(X;\psi)\otimes\mathbb{Q} \xrightarrow{\cong} H_i(X;\widetilde{\mathbb{Q}}_\psi) \tag{2.1}$$

is bijective for all $i\in\mathbb{Z}$ (see [D], proposition 5.2). Here $\widetilde{\mathbb{Q}}_\psi$ denotes the rational coefficient system which is twisted like the orientation line bundle ξ_ψ of ψ (i.e., $w_1(\xi_\psi) = w_1(\psi)$).

LEMMA 2.2: *Let $\psi = \gamma + \psi_0$ be the sum of a c-dimensional vector bundle γ and a virtual vector bundle ψ_0. Then the involution $(-\mathrm{id}_\gamma)_*$ on $\Omega_*(X;\psi)\otimes\mathbb{Q}$ (induced by the antipodal map on γ, compare (1.3)) equals $(-1)^c\cdot$ identity.*

Proof: In the rational setting the involution depends only on its effect on orientations (by (2.1)). ∎

In particular, consider the boundary endomorphism ∂_k on $\Omega_k(P(\alpha);\phi)$ (cf. (I.7) and (1.3)); when tensored with the identity map on $\mathbb{Q}$, it takes the form

$$\partial_k^{\mathbb{Q}} = \pm((-\operatorname{id}_\beta)_* - \operatorname{id}) \otimes \operatorname{id}_{\mathbb{Q}} = \pm((-1)^b - 1) \cdot \text{identity}.$$

PROPOSITION 2.3: (i) *The image of ∂_k is infinite if and only if b is odd and $\Omega_k(P(\alpha);\phi) \otimes \mathbb{Q} \neq 0$.*

(ii) *The kernel of ∂_k is infinite if and only if b is even and $\Omega_k(P(\alpha;\phi) \otimes \mathbb{Q} \neq 0$. (This is also equivalent to $\operatorname{coker} \partial_k$ being infinite.)*

Proof: The group $\Omega_k(P(\alpha);\phi)$ is finitely generated since it equals a stable homotopy group of a (finite) Thom complex. Thus tensoring with $\mathbb{Q}$ preserves exactness. We obtain the commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & (\ker \partial_k)\otimes\mathbb{Q} & \longrightarrow & \Omega_k(P(\alpha);\phi)\otimes\mathbb{Q} & \longrightarrow & (\operatorname{im}\partial_k)\otimes\mathbb{Q} & \to & 0 \\ & & & & & \searrow^{((-1)^b-1)\cdot} & \downarrow & & \\ & & & & & & \Omega_k(P(\alpha);\phi)\otimes\mathbb{Q} & & \end{array}$$

with exact horizontal sequence. Now, $\operatorname{im}\partial_k$ is infinite precisely if $(\operatorname{im}\partial_k)\otimes\mathbb{Q} \neq 0$ or, equivalently, $((-1)^b - 1)\cdot\Omega_k(P(\alpha);\phi)\otimes\mathbb{Q} \neq 0$. Claim (i) follows, and so does claim (ii) by similar arguments. ∎

It remains to decide when $\Omega_k(P(\alpha);\phi)\otimes\mathbb{Q}$ is nontrivial. For this we may use the rational Hurewicz isomorphism (2.1) and standard arguments from homology theory with twisted coefficients. But it is more in keeping with our geometric approach to consider (the rational analoga of) the exact Gysin sequences

(2.4)
$$\cdots \xrightarrow{st_j} \Omega_j(P^\infty \times N;\phi) \xrightarrow{\tau_j} \Omega_{j-a}(P^\infty \times N; \lambda\otimes\beta - TN) \to \Omega_{j-1}(P(\alpha);\phi) \to \cdots$$

and, for any virtual coefficient bundle ψ,

(2.5)
$$\cdots \to \Omega_j(P^\infty \times N;\psi) \xrightarrow{\Delta_j} \Omega_{j-1}(P^\infty \times N;\psi+\lambda) \to \Omega_{j-1}(S^\infty \times N;\pi^*(\psi)) \to \cdots$$

(compare [D], I.6 or, e.g., [K 3], 2.1). Here τ_j and Δ_j are defined by taking zero sets of generic sections s in (the pullback of) $\lambda\otimes\alpha$ and λ, respectively. Since we may use the section $-s$ instead, we have, e.g., that $\tau_j = (-\operatorname{id}_{\lambda\otimes\alpha})_* \circ \tau_j$ and hence, by Lemma 2.2,

$$\tau_j^{\mathbb{Q}} := \tau_j \otimes \operatorname{id}_{\mathbb{Q}} = (-1)^a \tau_j^{\mathbb{Q}}; \tag{2.6}$$

thus $\tau_j^{\mathbb{Q}} = 0$ if a is odd.

For the same reason (and since $S^\infty \sim *$), $\Delta_j \otimes \mathrm{id}_{\mathbb{Q}}$ is always trivial and the rational version of the long sequence (2.5) splits into the short exact sequences

$$0 \to \Omega_j(P^\infty \times N; \psi + \lambda) \otimes \mathbb{Q} \to \Omega_j(N; \psi|N) \otimes \mathbb{Q} \underset{\mathrm{proj}_*}{\overset{\mathrm{incl}_*}{\rightleftarrows}} \Omega_j(P^\infty \times N; \psi) \otimes \mathbb{Q} \to 0 .$$

Now decompose

$$\text{(2.7)} \qquad w_1(\psi) = c w_1(\lambda) + d \in H^1(P^\infty \times N; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus H^1(N; \mathbb{Z}_2).$$

If $c = 0$, i.e., if the orientation bundle of ψ (and hence $\Omega_j(P^\infty \times N; \psi) \otimes \mathbb{Q}$) does not involve λ (compare (2.1)), then the rational homomorphism incl_* above is bijective; indeed, the projection to N induces an inverse. Moreover, $\Omega_j(P^\infty \times N; \psi + \lambda) \otimes \mathbb{Q} = 0$ in this case. We obtain

LEMMA 2.8:

$$\Omega_*(P^\infty \times N; \psi) \otimes \mathbb{Q} \cong \begin{cases} \Omega_*(N; \psi|N) \otimes \mathbb{Q} \cong H_*(N; \widetilde{\mathbb{Q}}_{\psi|N}) & \text{if } c = 0; \\ 0 & \text{if } c \neq 0 \text{ in (2.7).} \end{cases}$$

This, together with (2.4), implies

LEMMA 2.9:

$$\Omega_j(P(\alpha); \phi) \otimes \mathbb{Q} \cong \begin{cases} \Omega_j(N; \eta) \otimes \mathbb{Q} \cong H_j(N; \widetilde{\mathbb{Q}}_\eta) & \text{if } a \equiv b \not\equiv 0(2); \\ H_{j-a+1}(N; \widetilde{\mathbb{Q}}_{\beta - TN}) & \text{if } a \not\equiv b \equiv 0(2); \\ 0 & \text{if } a \not\equiv b \not\equiv 0(2); \\ \ker \tau_j^{\mathbb{Q}} \oplus \operatorname{coker} \tau_{j+1}^{\mathbb{Q}} & \text{if } a \equiv b \equiv 0(2). \end{cases}$$

Proof: If $a \not\equiv 0(2)$, then $\tau_*^{\mathbb{Q}} = 0$ and the rational version of 2.4 shows that $\Omega_j(P(\alpha); \phi) \otimes \mathbb{Q}$ is isomorphic to

$$\Omega_{j-a+1}(P^\infty \times N; \lambda \otimes \beta - TN) \otimes \mathbb{Q} \oplus \Omega_j(P^\infty \times N; \phi) \otimes \mathbb{Q};$$

but by Lemma 2.8 one of these summands vanishes while (in view of (2.1)) the other summand can be expressed as indicated.

If a is even, the claims made above follow again from our rational sequence; in particular, if also $b \not\equiv 0(2)$ two of our three groups vanish by Lemma 2.8 and therefore so does the third. ∎

The corollary to Theorem D in the introduction follows now directly from Theorems A, B and D as well as from the results (2.1), Proposition 2.3 and Lemmas 2.8 and 2.9 discussed above.

With nearly no extra work, the methods of this section lead also to a better understanding of the transition from "strong" to "weak, stable" invariants.

PROPOSITION 2.10: *Given any $j \in \mathbb{Z}$, the forgetful homomorphism*

$$\operatorname{forg}_j : \Omega_j(P(\alpha); \phi) \longrightarrow \overline{\Omega}_j(P^\infty \times N; \phi)$$

has an infinite kernel if and only if b is even and $H_{j-a+1}(N; \widetilde{\mathbb{Q}}_{\beta - TN}) \neq 0$ (and, in case a is also even, $\tau_{j+1}^{\mathbb{Q}}$ is not onto, cf. (I.11)).

An analoguous result holds for $\widetilde{\operatorname{forg}}_j$ (and $\widetilde{\alpha}, \widetilde{\phi}$ and $\widetilde{N}$).

Proof: forg_j is the composite of the "stabilizing" (or "inclusion") map st_j (cf. 2.4) and the homomorphism $\overline{f}_j$ which retains only orientation information. Now the rational Hurewicz isomorphism $\mu_j^{\mathbb{Q}}$ (cf. 2.1) factors through $\overline{f}_j^{\mathbb{Q}} = \overline{f}_j \otimes \mathrm{id}_{\mathbb{Q}}$ and so $\overline{f}_j^{\mathbb{Q}}$ is injective. Therefore, forg_j has infinite kernel if and only if $\ker(st_j \otimes \mathrm{id}_{\mathbb{Q}}) \neq 0$. Our claim follows again from the rational version of the sequence (2.4). ∎

§3. The antipodality obstructions $v_i(\alpha, \beta)$

In this section we want to get some hold on the "weak, stable" obstructions $\operatorname{forg}_*(\omega(\alpha,\beta)), \widetilde{\operatorname{forg}}_*(\omega(\widetilde{\alpha},\beta))$ and $\operatorname{forg}_*(d(u,-u))$ (cf. (I.7) and (I.13)). For this purpose we calculate the values of their images under the mod 2 Hurewicz homomorphisms

$$\text{(3.1)} \qquad \begin{aligned} \mu_2 &: \overline{\Omega}_*(P^\infty \times N; \phi) \longrightarrow H_*(P^\infty \times N) \quad \text{and} \\ \mu_2 &: \overline{\Omega}_*(P^\infty \times \widetilde{N}; \widetilde{\phi}) \longrightarrow H_*(P^\infty \times \widetilde{N}). \end{aligned}$$

We encounter —besides standard Stiefel–Whitney obstructions— also the highly interesting antipodality obstructions $v_i(\alpha,\beta) \in H^i(N)$ defined in (I.14).

THEOREM 3.2:

(i) $\mu_2(\overline{\omega}(\alpha,\beta)) = 0$ *if and only if* $w_i(\beta - \alpha) = 0$ *for* $i > b - a$;

(ii) $\mu_2(\overline{\omega}(\widetilde{\alpha},\beta)) = 0$ *iff both* $w_i(\beta-\alpha)$ *and* $v_{i-1}(\alpha,\beta)$ *vanish for all* $i > b - a$;

(iii) *when a monomorphism* $u : \alpha \hookrightarrow \beta$ *exists, the antipodality obstruction* $\mu_2(\overline{d}(u,-u))$ *is independent of* u *and vanishes precisely if* $v_i(\alpha,\beta) = 0$ *for* $i \geq b - a$.

Proof: Let $x \in H^1(P^\infty)$ and $y = w_1(\xi) \in H^1(S^1)$ denote the canonical generators.

For the first claim represent $\omega(\alpha,\beta)$ by the zero set data $(Z \subset P(\alpha), \overline{g})$ as in (I.1) Also pick a complementary vector bundle $\alpha^\perp$ over N such that $\alpha \oplus \alpha^\perp = \mathbb{R}^L \times N$ for some $L \gg 0$. The projection onto this summand defines a generic section of $\mathrm{Hom}(\lambda, \alpha^\perp) \cong \lambda \otimes \alpha^\perp$ over $P^{L-1} \times N$ with zero set $P(\alpha)$. Thus Z

occurs as the zero set of a generic section of the $(b+L-a)$-dimensional vector bundle $\lambda\otimes(\beta\oplus\alpha^\perp)$ over $P^{L-1}\times N$.

Now, given $j\geq 0$ and $c\in H^{n+a-b-j-1}(N)$, we calculate by standard techniques (see, in particular, formulas (0.18) in [K 3] and (9.9) in [K 1])

$$
\begin{aligned}
x^j c(\mu_2(\overline{\omega}(\alpha,\beta))) &= (x^j c \mid Z)[Z]\\
&= x^j c w_{b-a+L}(\lambda\otimes(\beta\oplus\alpha^\perp))[P^{L-1}\times N]\\
&= \sum_{i\geq 0} x^{i+j} c w_{b-a+L-i}(\beta\oplus\alpha^\perp)[P^{L-1}\times N]\\
&= c w_{b-a+j+1}(\beta-\alpha)[N].
\end{aligned}
\tag{3.3}
$$

Clearly, $\mu_2(\overline{\omega}(\alpha,\beta))=0$ precisely if all these characteristic numbers vanish, i.e., if $w_{b-a+j+1}(\beta-\alpha)=0$ for $j\geq 0$. This proves our first claim.

Similarly, we conclude that $\mu_2(\overline{\omega}(\widetilde{\alpha},\beta))$ is trivial if and only if $w_i(\beta-\widetilde{\alpha})=0$ for all $i>b-a$. Thus let us compute these Stiefel–Whitney classes.

According to the splitting principle (cf. [H], 4.4.3) we can calculate the total class

$$w(\widetilde{\alpha})=w(\xi\otimes\alpha)\in H^*(S^1\times N)$$

and its inverse as if α were a direct sum of line bundles. Thus formally we can write

$$w(\alpha)=\prod_{i=1}^{a}(1+z_i),\quad w(\widetilde{\alpha})=\prod_{i=1}^{a}(1+z_i+y)$$

and therefore (since $y^2=0$)

$$
\begin{aligned}
\overline{w}(\widetilde{\alpha}) &\equiv \prod_{i=1}^{a}(1+z_i+y+z_i^2+(z_i+y)z_i^2+z_i^4+\cdots\\
&=\prod_{i=1}^{a}\Big(\sum_{j\geq 0} z_i^{2j}[(1+z_i)+y]\Big)\\
&=\prod_{i=1}^{a}(1+z_i)^{-2}\Big[\Big(\prod_{i=1}^{a}(1+z_i)\Big)+y\Big(\sum_{i=1}^{a}(1+z_1)\cdots\widehat{(1+z_i)}\cdots(1+z_a)\Big)\Big]\\
&=\overline{w}(\alpha)+y\overline{w}(\alpha)^2\sum_{i=1}^{a}\Big(\sum_{1\leq j_1<j_2<\cdots<j_k\leq a;\,\mathrm{all}\, j_s\neq i} z_{j_1}\cdot\ldots\cdot z_{j_k}\Big)\\
&=\overline{w}(\alpha)+y\overline{w}(\alpha)^2\cdot\sum_{k=0}^{a}(a-k)w_k(\alpha);
\end{aligned}
$$

here a summand $z_{j_1}\ldots z_{j_k}$ gets counted once for every $i\notin\{j_1,\ldots j_k\}$, i.e., altogether $a-k$ times.

We obtain for $j \in \mathbb{Z}$

$$\begin{aligned}\overline{w}_j(\tilde{\alpha}) =&\overline{w}_j(\alpha) + y\sum_{i=0}^{\infty}\overline{w}_i(\alpha)^2 \cdot (a-j+2i+1)w_{j-2i-1}(\alpha)\\ =&\overline{w}_j(\alpha) + (a+j-1)y\overline{w}_{j-1}(\alpha)\end{aligned} \tag{3.4}$$

and finally for all natural numbers i

$$\begin{aligned}w_i(\beta-\tilde{\alpha}) =& \sum_j(\overline{w}_j(\alpha) + (a+j-1)y\overline{w}_{j-1}(\alpha))w_{i-j}(\beta)\\ =&w_i(\beta-\alpha) + y\cdot v_{i-1}(\alpha,\beta)\end{aligned} \tag{3.5}$$

(compare (I.14)). This proves the second claim in Theorem 2.2 and explains why the v-invariants play such a central role.

Claim (iii) now follows easily from the injectivity of the homomorphism

$$\text{incl}_*: H_*(P^\infty \times N) \longrightarrow H_*(P^\infty \times S^1 \times N),$$

from its compatibility with $\mu_2 \circ \widetilde{\text{forg}}_*$ and from relation (I.9). Indeed, if u exists, then $\mu_2 \circ \text{forg}_*(d(u,-u)) = 0$ precisely if for $i > b-a$ the Stiefel–Whitney class $w_i(\beta - \tilde{\alpha}) = yv_{i-1}(\alpha,\beta)$ vanishes.

A second, more direct proof of claim (iii) can be based on (the discussion of) Proposition 1.5. Using the same standard techniques as in (3.3) we obtain for

$$\pm d(u,-u) = \omega(\underset{\sim}{\mathbb{R}}, (\mathbb{R}s_u)^\perp) = [Z \subset P(\alpha), \overline{g}]$$

and for all $j \geq 0$ and $c \in H^{n+a-b-j}(N)$

$$\begin{aligned}x^j c(\mu_2(\overline{d}(u,-u))) =& (x^j c \mid Z)[Z]\\ =&x^j c w_{b-1}(\lambda\otimes\beta)[P(\alpha)]\\ =&x^j c w_{b-1}(\lambda\otimes\beta)w_{L-a}(\lambda\otimes\alpha^\perp)[P^{L-1}\times N]\\ =&x^j c\Big(\sum_{i\geq 0}(b-i)x^{b-1-i}w_i(\beta)\Big)\cdot\sum_{k\geq 0}x^{L-a-k}w_k(\alpha^\perp)[P^{L-1}\times N]\\ =&c\sum_{k\geq 0}(a+k-j)\overline{w}_k(\alpha)w_{b-a+j-k}(\beta)[N]\\ =&c(v_{b-a+j}(\alpha,\beta) - jw_{b-a+j}(\beta-\alpha))[N].\end{aligned}$$

Note that the last summand to the right vanishes due to the existence of u. Thus again we see that $\mu_2(\overline{d}(u,-u))$ is trivial if and only if $v_{b-a+j}(\alpha,\beta) = 0$ for all $j \geq 0$. ∎

Next, let us study some of the formal properties of the invariants

$$v_i(\alpha,\beta) = \sum_{j\geq 0}(a+j)\overline{w}_j(\alpha)w_{i-j}(\beta), \quad i\in\mathbb{Z}, \tag{3.6}$$

defined for any vector bundles α and β over N (where a denotes the dimension of α). Clearly we have

$$\begin{aligned} v_i(\alpha\oplus \underset{\sim}{\mathbb{R}},\beta\oplus\underset{\sim}{\mathbb{R}}) &= \sum(a+j+1)\overline{w}_j(\alpha)w_{i-j}(\beta) \\ &= v_i(\alpha,\beta)+w_i(\beta-\alpha). \end{aligned} \tag{3.7}$$

Therefore $v_i(\alpha,\beta)$ does not only depend on the K-theoretical difference $\beta-\alpha$. But there are further interesting interrelations with the Stiefel–Whitney classes of $\beta-\alpha$ such as the following "mixed Wu formula".

LEMMA 3.8: *For all $i\in\mathbb{Z}$*

$$Sq^1(v_i(\alpha,\beta)) = w_1(\beta-\alpha)v_i(\alpha,\beta)+iv_{i+1}(\alpha,\beta)+aw_{i+1}(\beta-\alpha).$$

Proof: Using simple special cases of the standard Wu and Cartan formulas (see, e.g., [MS], §8 and in particular problem 8–A), we obtain

$$\begin{aligned} Sq^1(v_i(\alpha,\beta)) &= \sum(a+j)(\overline{w}_1(\alpha)\overline{w}_j(\alpha)w_{i-j}(\beta)+(j+1)\overline{w_{j+1}}(\alpha)w_{i-j}(\beta)) \\ &\quad+\sum(a+j)(w_1(\beta)\overline{w}_j(\alpha)w_{i-j}(\beta)+(i-j+1)\overline{w}_j(\alpha)w_{i-j+1}(\beta)) \\ &= w_1(\beta-\alpha)v_i(\alpha,\beta)+\sum((a+j-1)j+(a+j)(i-j+1))\overline{w}_j(\alpha)w_{i-j+1}(\beta) \end{aligned}$$

and the lemma follows since

$$a+((a+j-1)j+(a+j)(i-j+1)) = (a+j)\cdot i. \qquad \blacksquare$$

Example 3.9: $a=1$. If α is a line bundle and β has dimension b, then for all $i\in\mathbb{Z}$

$$v_i(\alpha,\beta) = \sum_{k\geq 0} w_1(\alpha)^{2k}w_{i-2k}(\beta) = w_i(\beta-2\alpha)$$

whereas $w_i(\beta-\alpha)=\sum_{j\geq 0}w_1(\alpha)^j w_{i-j}(\beta)$. In particular,

$$v_{b-1}(\alpha,\beta)=w_{b-1}(\alpha\otimes\beta) \quad\text{and}\quad w_b(\beta-\alpha)=w_b(\alpha\otimes\beta)$$

(use, e.g., [K 1], 9.9, again). These invariants can have a precise geometric significance already in very simple situations where they just measure 0-dimensional singularities. Assume, e.g., that $b=n\geq 1$. Given any monomorphism $u\colon \alpha^1\hookrightarrow\beta^n$, it follows from Theorem 3.2 that u is *concordant* (i.e., regularly homotopic in $\beta \oplus \underset{\sim}{\mathbb{R}}$) to $-u$ if and only if

$$w_n(\beta - 2\alpha) = v_n(\alpha, \beta \oplus \underset{\sim}{\mathbb{R}}) = 0;$$

this condition holds automatically when $w_1(\alpha \otimes \beta - TN) = w_1(\alpha)$ and hence the concordance obstruction is an integer which lies in the image of $\overline{\Delta}_\alpha$ and therefore must vanish (compare formula (4.2′) below). The special case $\beta = TN$ has been discussed in detail in [K 0].

§4. The weak stable obstructions

In this section we study the invariants $\overline{d}(u, -u) = \text{forg}_*(d(u, -u))$, $\overline{\omega}(\widetilde{\alpha}, \beta) = \widetilde{\text{forg}}_*(\omega(\widetilde{\alpha}, \beta))$ and $\overline{\omega}(\alpha, \beta) = \text{forg}_*(\omega(\alpha, \beta))$. In particular, we prove Theorem E and Proposition F of the introduction.

For $Y = N$ (or $\widetilde{N}$) and $\varphi = \phi$ (or $\widetilde{\phi}$), let

$$(4.1) \qquad \overline{\Delta}\colon \overline{\Omega}_j(P^\infty \times Y; \varphi) \longrightarrow \overline{\Omega}_{j-1}(P^\infty \times Y; \varphi + \lambda), \quad j \in \mathbb{Z},$$

be the homomorphism obtained by taking zero sets of generic sections in (the pullback — under the projection to P^∞ — of) the universal line bundle λ (compare [K 3], (1.3)).

Then we have

$$(4.2) \qquad \overline{\Delta}(\overline{\omega}(\alpha, \beta)) = \overline{\omega}(\alpha, \beta \oplus \underset{\sim}{\mathbb{R}}), \quad \overline{\Delta}(\overline{\omega}(\widetilde{\alpha}, \beta)) = \overline{\omega}(\widetilde{\alpha}, \beta \oplus \underset{\sim}{\mathbb{R}})$$

and, if $i\colon \beta \hookrightarrow \beta \oplus \mathbb{R}$ denotes the inclusion,

$$(4.2') \qquad \overline{\Delta}(\overline{d}(u, -u)) = \overline{d}(i \cdot u, -i \cdot u)$$

for every monomorphism $u\colon \alpha \hookrightarrow \beta$. This follows essentially by construction (compare, e.g., the discussion preceding (I.1)).

Now consider the case $a \not\equiv b(2)$. Then

$$w_1(\phi) = w_1(\lambda) + w_1(\beta - \alpha - TN)$$

and the second projection $\pi\colon P^\infty \times N \to N$ induces the isomorphism

$$(4.3) \qquad \pi_*\colon \overline{\Omega}_k(P^\infty \times N; \phi) \xrightarrow{\cong} \mathfrak{N}_k(N)$$

onto unoriented bordism (see, e.g., proposition 1.3 in [K 3]). Adopting the notation of (I.2) we conclude therefore that $\overline{\omega}(\alpha, \beta) = 0$ if and only if all Whitney numbers of $[Z, \pi|] \in \mathfrak{N}_*(N)$ vanish (see [CF], 17.2). But this holds precisely if $\mu_2(\overline{\omega}(\alpha, \beta)) = 0$ since $w(Z) = w^{-1}(\phi)|Z$ (by (I.1)) and, on the other hand,

$$w_1(\lambda)|Z = w_1(Z) + w_1(\beta - \alpha - TN)|Z.$$

Theorem 3.2 now implies those statements in Theorem E which concern $\overline{\omega}(\alpha, \beta)$. The remaining claims follow in the same way, and so does Proposition F: just note that the projection from $\widetilde{N} = S^1 \times N$ onto N induces a left inverse of $\overline{\text{incl}}_*$.

Next we turn to the remaining case $a \equiv b(2)$. Then $w_1(\phi) = w_1(\beta - \alpha - TN) = w_1(\eta)$ and we get the homomorphisms

$$\overline{\Omega}_*(P^\infty \times N; \phi) \xrightarrow{\overline{\pi}_*} \overline{\Omega}_*(N; \eta) \xrightarrow{\mu} H_*(N; \widetilde{\mathbb{Z}}_\eta) \tag{4.4}$$

(where $\widetilde{\mathbb{Z}}_\eta$ denotes the integer coefficient system which is twisted like the orientation line bundle of η; compare (I.12) and (I.15)). The isomorphisms in (I.7) are given by $\overline{\pi}_*$ and $\pi_* \circ \overline{\Delta}$ (compare (4.4), (4.1) and (4.3)) or their analogues for $\widetilde{N}$. Now $\overline{\Delta}$ maps our weak stable invariants of the pair (α, β) to the corresponding invariants for $(\alpha, \beta \oplus \underset{\sim}{\mathbb{R}})$ (cf. (4.2) and (4.2′)). Again, since $a \not\equiv \dim(\beta \oplus \underset{\sim}{\mathbb{R}})(2)$ and since $v_i(\alpha, \beta \oplus \underset{\sim}{\mathbb{R}}) = v_i(\alpha, \beta)$ the previous discussion establishes all claims in Theorem E and Proposition F concerning the second, $\mathfrak{N}_*$-components of our weak stable obstructions.

Thus consider their images under $\overline{\pi}_*$ and the Hurewicz homomorphism μ (cf. (4.4)). We know from [K 1], 5.3 and from Poincaré duality that

$$\mu \circ \overline{\pi}_*(\overline{d}(u, -u)) = 0 \quad \text{iff } \overline{c}(u, -u) = 0, \quad \text{and}$$
$$\mu \circ \overline{\pi}_*(\overline{\omega}(\alpha, \beta)) = 0 \quad \text{iff } \overline{c}(\alpha, \beta) = 0.$$

Now assume that for some dimension j every element of $\overline{\Omega}_j(N; \eta)$ is entirely determined by its image under μ, together with its Whitney numbers (and its Pontryagin numbers if $w_1(\eta) = 0$); this holds, e.g., if $j \leq 4$ or if condition (iii) in Theorem E is satisfied (see [O], 0.13, and [CF], 17.5). Then, given any element $\omega = [Z, g, \overline{g}]$ of $\Omega_j(P(\alpha); \phi)$, its image $\overline{\omega} := \text{forg}_*(\omega)$ vanishes precisely if $\overline{\Delta}(\overline{\omega}) = 0$ and $\mu \circ \overline{\pi}_*(\overline{\omega}) = 0$. Indeed, the stable vector bundle isomorphism $\overline{g}$ implies again that $TZ = -g^*(\phi) \in \widetilde{K}(Z)$ and therefore $w(Z) = g^*(w(\phi))^{-1}$ (and $p(Z) \equiv g^*(p(\phi))^{-1} \equiv (\pi \circ g)^*(p(\eta))^{-1}$ modulo cohomology classes of order 2); hence the characteristic numbers of $\overline{\pi}_*(\overline{\omega})$ factor through $\overline{\Delta}(\overline{\omega})$ and $\mu \circ \overline{\pi}_*(\overline{\omega})$. In the special cases when ω equals $\omega(\alpha, \beta)$ or $d(u, -u)$, this implies the corresponding vanishing criteria in Theorem E.

The same argument applies also to $\omega(\widetilde{\alpha}, \beta)$. Moreover, note that the boundary endomorphism $\overline{\partial}_*$ in diagram (I.7) is given by multiplication with $\pm(1 - (-1)^b)$

(compare (1.3)). Thus in the case $a \equiv b \equiv 1(2)$ (when $w_1(\widetilde{\eta}) \neq w_1(\eta)$) the vertical exact sequence to the right hand side in (I.7) contains the exact sequence

$$\begin{array}{ccccccccc} \cdots \to & \overline{\Omega}_k(N;\eta) & \xrightarrow{\partial_* = \pm 2\cdot} & \overline{\Omega}_k(N;\eta) & \xrightarrow{\overline{\mathrm{incl}}_*} & \overline{\Omega}_k(S^1 \times N; \widetilde{\eta}) & \xrightarrow{\pitchfork} & \overline{\Omega}_{k-1}(N;\eta) & \to \cdots \\ & & & & \searrow & \downarrow & & & \\ & & & & & \mathfrak{N}_k(N) & & & \end{array} \tag{4.5}$$

as a direct summand and the vanishing criterion for $\overline{\omega}(\widetilde{\alpha}, \beta)$ follows as before (use also the generalized Rochlin theorem 16.2 in [CF] and Lemma 3.8). In the case $a \equiv b \equiv 0(2)$ we have: $w_1(\widetilde{\eta}) = w_1(\eta), \overline{\partial}_* = 0$ and $\overline{\mathrm{incl}}_*$ is even split injective, again with a left inverse induced by the obvious projection. Thus $\overline{\omega}(\widetilde{\alpha}, \beta)$ can be decomposed into $\overline{\omega}(\alpha, \beta)$ and a second component which is always well defined and which agrees with $\overline{d}(u, -u)$ whenever u exists.

Finally, observe that by (4.2) and the previous discussion

$$\overline{\omega}(\alpha, \beta) = \overline{\omega}(\alpha \oplus \underset{\sim}{\mathbb{R}}, \beta \oplus \underset{\sim}{\mathbb{R}}) = \overline{\Delta}(\overline{\omega}(\alpha \oplus \underset{\sim}{\mathbb{R}}, \beta))$$

is the image of an element of order 2 and so is $\overline{\omega}(\widetilde{\alpha}, \beta)$. This completes the proof of Theorem E and Proposition F. ∎

§5. The singularity sequences and some examples of injectivity and existence criteria

The singularity approach allows one not only to decide existence and classification questions concerning (antipodal) monomorphisms, but it also offers powerful tools for the study of low-dimensional normal bordism groups (see, e.g., the exact sequences in [K 1], 9.3, and [K 3], 3.1). We will use them in this section to obtain a few concrete applications and illustrations of the general results developed so far. In particular, we will be interested (in some sample cases) in injectivity criteria concerning ∂_* (and hence $d(u, -u)$) as well as forg. We will also spell out a few specific existence conditions for antipodal monomorphisms even in cases where higher order obstructions have to be determined by rather involved geometric arguments. This will then be used to discuss Example G of the introduction.

For most calculations it is important to know the first two Stiefel–Whitney classes of the coefficient bundles ϕ and $\widetilde{\phi}$ (cf. (I.3) and (I.6)). By standard computations we obtain in $H^*(P^\infty \times S^1 \times N)$

$$(5.1)\qquad \begin{aligned} w_1(\widetilde{\phi}) &= (a+b)x + w_1(\eta) + ay \quad \text{and} \\ w_2(\widetilde{\phi}) &= \binom{b-a}{2}x^2 + x(w_1(N) + (a+b+1)w_1(\eta)) + w_2(\eta) \\ &\quad + y(abx + w_1(\alpha) + aw_1(\eta)) \end{aligned}$$

(where $x = w_1(\lambda)$ and y generate $H^1(P^\infty)$ and $H^1(S^1) \subset H^1(S^1 \times N)$, resp.; cf. also (I.12)). Restriction to $P(\widetilde{\alpha})$ and to $P(\alpha) \subset P^\infty \times N \subset P^\infty \times S^1 \times N$ yields the desired formulas, e.g., also for ϕ (where we just have to drop the y-term).

We will illustrate our theory in the case $(a,b) = (3, n+1)$.

First consider the special setting where α and β are (real) vector bundles of dimensions 3 and $n+1$ over complex projective space $N^n = \mathbb{C}P(q)$. Denote by $\lambda_1 = \lambda_{\mathbb{C}} \subset \underset{\sim}{\mathbb{C}}^{q+1}, \lambda_{-1} = \overline{\lambda}_{\mathbb{C}}$ and $\lambda_1^\perp$ the canonical complex line bundle over $\mathbb{C}P(q)$, its complex conjugate and its complement in $\underset{\sim}{\mathbb{C}}^{q+1}$, respectively. Also, recall the well-known complex isomorphism

$$(5.2)\qquad T\mathbb{C}P(q) \cong \underset{\sim}{\mathrm{Hom}}_{\mathbb{C}}(\lambda_1, \lambda_1^\perp) \cong \lambda_{-1} \otimes_{\mathbb{C}} \lambda_1^\perp$$

(see [MS], p. 169). Finally, fix the generators $z = c_1(\lambda_1)$ and $z_2 = w_2(\lambda_1)$ of $H^2(N;\mathbb{Z})$ and of $H^2(N) := H^2(N;\mathbb{Z}_2)$. Then $w_1(\phi) = 0$ and

$$(5.3)\qquad w_2(\phi) = (q-1)x^2 + w_2(\eta)$$

where $w_2(\eta) = (q+1)z_2 + w_2(\alpha) + w_2(\beta)$.

As in [K 3], 3.1, the singularity sequence in theorem 9.3 of [K 1] gives rise to the exact sequences

$$(5.4)\qquad \mathfrak{N}_2(N) \xrightarrow{(\sigma j_3, \overline{\tau})} \Theta_1 \oplus \mathbb{Z}_2 \xrightarrow{\delta - \widehat{\delta}} \Omega_2(P(\alpha);\phi) \xrightarrow{\mathrm{forg}_2} \Omega_2(N) \xrightarrow{w_2(\eta)} \mathbb{Z}_2$$

and

$$(5.4')\qquad \mathbb{Z}_2 \xrightarrow{\delta_1'} \Theta_1 := \Omega_1(P^\infty \times N \times BO(2); \phi + \Gamma) \xrightarrow{f_1'} H_1(P^\infty) = \mathbb{Z}_2 \to 0$$

where f_1' is bijective if and only if $w_2(\phi) \neq 0$. Here we use canonical isomorphisms

$$(5.5)\qquad \begin{aligned} &\overline{\Omega}_3(P^\infty \times N; \phi) \cong \mathfrak{N}_2(N) \cong H_2(N) \oplus \mathfrak{N}_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ &\text{and} \\ &\overline{\Omega}_2(P^\infty \times N; \phi) \cong \Omega_2(N) \cong H_2(N;\mathbb{Z}) \xrightarrow[\cong]{z} \mathbb{Z}. \end{aligned}$$

The first homomorphism in (5.4) is determined by standard calculations (using, e.g., [K 3], 0.18, 1.6 and 2.2) as follows: for any bordism class $s = [g\colon S \to N] \in \mathfrak{N}_2(N)$ we have

$$(5.6)\qquad \begin{aligned} f_1' \circ \sigma j_3(s) =&(q+1)w_1(S)^2[S] + g^*(w_2(\eta))[S];\\ \overline{\tau}(s) =&w_1(S)^2[S] + g^*(w_2(\alpha))[S]. \end{aligned}$$

In particular,

$$(5.7)\qquad h := f_1' \circ \sigma j_3 + (q+1)\overline{\tau}\colon \mathfrak{N}_2(N) \longrightarrow \mathbb{Z}_2$$

is obtained by evaluating $w_2(\eta) + (q+1)w_2(\alpha)$.

Now we are ready to analyze the endomorphism ∂_2 of $\Omega_2(P(\alpha);\phi)$ (cf. (I.7)) which plays such a central role in the classification of monomorphisms — antipodal or not (see Theorem D).

Proposition 5.8: *Let α^3 and β^{2q+1} be vector bundles over $N = \mathbb{C}P(q), q > 1$. The following conditions are equivalent:*

(i) *∂_2 is injective on $\Omega_2(P(\alpha);\phi)$;*
(ii) forg_2 *is injective on $\Omega_2(P(\alpha);\phi)$; and*
(iii) *$qw_2(\alpha) + w_2(\beta) + w_2(N) \neq 0$.*

If these conditions do not hold, then $\ker \partial_2 \cong \mathbb{Z}_2$; moreover, $\ker \mathrm{forg}_2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or to $\mathbb{Z}_2$ according as $w_2(\phi)$ vanishes or not, respectively.

In any case, forg_2 *is onto if and only if $w_2(\alpha) + w_2(\beta) + w_2(N) = 0$.*

Proof: Since b is odd, $\overline{\partial}_2$ is multiplication by ± 2 and hence injective on $\overline{\Omega}_2(P^\infty \times N;\phi) \cong \mathbb{Z}$ (compare (I.7), (1.3) and (5.5)). Therefore $\ker \partial_2 \subset \ker \overline{\partial}_2 \circ \mathrm{forg}_2 = \ker \mathrm{forg}_2$.

If condition (iii) holds or, equivalently, $h \not\equiv 0$ (cf. (5.7)) then $(f_1' \circ \sigma j_3, \overline{\tau})$ maps *onto* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$; in particular, the homomorphism $f_1' \circ \sigma j_3$ (which evaluates $w_2(\phi)$, cf. [K 1], 9.3) is nontrivial and therefore so is $w_2(\phi)$. Thus f_1' is an isomorphism (cf. (5.4′)) and forg_2 and hence ∂_2 must be injective (cf. (5.4)).

If $h \equiv 0$ but still $w_2(\phi) \neq 0$, then the image of $(\sigma j_3, \overline{\tau})$ in $\Theta_1 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is $\mathbb{Z}_2(q+1, 1)$. Hence $\ker \mathrm{forg}_2$ consists of two elements which are fixed by the involution $(-\mathrm{id}_\beta)_*$. It follows that $\partial_2 = \pm((-\mathrm{id}_\beta)_* - \mathrm{id}) = 0$ on $\ker \mathrm{forg}_2$ and $\ker \partial_2 = \ker \mathrm{forg}_2 = \mathbb{Z}_2$ in this case.

Finally, assume $w_2(\phi) = 0$. Then by an observation of C. Olk (see the footnote on p. 94 in [K 1]) $\sigma j_3 = 0$, and $\ker \mathrm{forg}_2 \cong \Theta_1$ has four elements (compare (5.4) and (5.4′)). Actually, replacing N by a single point in the whole preceding calculation we see that

$$\Theta_1 \cong \Omega_2(P^2;(b-a)\lambda) \cong \pi_{2q}(V_{a,b}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

(use also [K 1], 5.4, and Paechter's tables [P], p. 249). For similar reasons as outlined above, $(-\mathrm{id}_\beta)_*$ fixes $\ker f_1' = \mathbb{Z}_2 \cdot \delta_1'(1)$ so that ∂_2 vanishes on $\delta \circ \delta_1'(1)$. But if a bordism class $u = [S^1, g, \overline{g}] \in \Theta_1$ has a nontrivial image $f_1'(u)$ in $H_1(P^\infty) \cong \mathbb{Z}_2$, then $\delta(u)$ is not annihilated by ∂_2; in fact, a simple geometric argument involving a punctured cylinder $S^1 \times I$-small disk shows that $\partial_2(\delta(u)) = \delta \circ \delta_1'(1)$ (for more details see the proof of theorem 2.2 in [KMS]). Thus again $\ker \partial_2$ consists only of two elements.

The last claim in Proposition 5.8 follows directly from (5.4). ∎

Along the lines of the proof above, precise bijectivity conditions for forg can be worked out in much more general settings. In turn, they give rise to antipodality criteria. Indeed, easy diagram chasing proves the following

OBSERVATION 5.9: *Assume that* forg_k *in diagram* (I.7) *is an isomorphism for some integer* k. *Then*

$$(\widetilde{\mathrm{forg}}_k, ⋔)\colon \Omega_k(P(\widetilde{\alpha});\widetilde{\phi}) \longrightarrow \Omega_k(P^\infty \times \widetilde{N};\widetilde{\phi}) \oplus \Omega_{k-1}(P(\alpha);\phi)$$

is injective. In case $k = n+a-b$ *we have in particular:* $\omega(\widetilde{\alpha},\beta) = 0$ *if and only if both* $\omega(\alpha,\beta)$ *and* $\overline{\omega}(\widetilde{\alpha},\beta)$ *vanish.*

However, this isomorphism assumption (used for a purely algebraic argument) is usually far too restrictive. Additional geometric input allows us sometimes in much more general situations to decide precisely when the necessary conditions $\omega(\alpha,\beta) = 0$ and $\overline{\omega}(\widetilde{\alpha},\beta) = 0$ are also sufficient for the existence of an antipodal monomorphism or when (and which) extra obstructions come into play.

For an illustration, we discuss a setting where two higher order obstructions (with values in $\mathbb{Z}_2$) may play a role and where we can always eliminate one of them by a geometric trick in the spirit of Proposition 1.5.

THEOREM 5.10: *Let* $\alpha^3 = \alpha' \oplus \underset{\sim}{\mathbb{R}}$ *and* β^{n+1} *be vector bundles over a manifold* N *of even dimension* $n > 4$, *where* α' *is a complex line bundle and* $H_1(N) := H_1(N;\mathbb{Z}_2) = 0$. *Assume that* $\omega(\alpha,\beta)$ *and* $w_{n-2}(\beta-\alpha)$ *vanish.*

Then $\omega(\widetilde{\alpha},\beta) \neq 0$ *if and only if* $w_2(\eta) \equiv 0$ *on all elements of order 2 in* $H_2(N;\mathbb{Z})$, *but there exists* $\overline{e} \in H_2(N;\mathbb{Z})$ *and a monomorphism* $u\colon \alpha \hookrightarrow \beta$ *such that* $w_2(\eta)(\overline{e}) \neq 0$ *and* $2\overline{e}$ *is equal to* $\mu(\overline{\omega}(\underset{\sim}{\mathbb{R}}, u(\alpha)^\perp))$ *(i.e., to the Poincaré dual of the Euler class of the cokernel bundle* $u(\alpha)^\perp$ *of* u*).*

Proof: Here clearly $w_2(\eta) = w_2(\alpha)+w_2(\beta)+w_2(N)$ (cf. (I.12)). Also, $v_i(\alpha,\beta) = w_i(\beta-\alpha)$ for all $i \in \mathbb{Z}$ so that the assumption above is equivalent to $\omega(\alpha,\beta)$ and $\overline{\omega}(\widetilde{\alpha},\beta)$ being trivial (by Theorem E and Lemma 3.8).

Now consider the analoga of the sequences (5.4) and (5.4′) for N and $\widetilde{N} = S^1 \times N$. Since $w_2(\widetilde{\phi}) = xy + \cdots \neq 0$ (cf. (5.1)), $\widetilde{f}_1'$ is bijective and we obtain the commuting diagram of horizontal and vertical exact sequences

(5.11)

$$\begin{array}{ccccccccc}
 & & & & \Omega_2(P(\alpha);\phi) & \xrightarrow{\text{forg}_2} & \overline{e} \in \Omega_2(N) & \xrightarrow{w_2(\eta)} & \\
 & & & & \downarrow{\scriptstyle \partial_*} & & \downarrow{\scriptstyle \cdot 2} & & \\
\xrightarrow{(\sigma j_3,\overline{\tau})} & \Theta_1 \oplus \mathbb{Z}_2 & \longrightarrow & d \in & \Omega_2(P(\alpha);\phi) & \xrightarrow{\text{forg}_2} & \Omega_2(N) & \xrightarrow{w_2(\eta)} & \\
 & \downarrow{\scriptstyle (0,f_1')\oplus \text{id}_{\mathbb{Z}_2}} & & & \downarrow{\scriptstyle \text{incl}_2} & & \downarrow{\scriptstyle \overline{\text{incl}}_2} & & \\
\xrightarrow{(\widetilde{\sigma j_3},\widetilde{\overline{\tau}})} & H_1(S^1)\oplus H_1(P^\infty)\oplus\mathbb{Z}_2 & \longrightarrow & & \Omega_2(P(\widetilde{\alpha});\widetilde{\phi}) & \xrightarrow{\widetilde{\text{forg}_2}} & \overline{\Omega}_2(P^\infty\times\widetilde{N};\widetilde{\phi}) & \longrightarrow & \cdots \\
 & \downarrow{\scriptstyle \pitchfork'} & & & \downarrow{\scriptstyle \pitchfork} & & \downarrow{\scriptstyle \overline{\pitchfork}} & & \\
\xrightarrow{w_2(\eta)} & \mathbb{Z}_2 & \longrightarrow & & \Omega_1(P(\alpha);\phi) & \xrightarrow{\text{forg}_1} & \Omega_1(N)\oplus\mathfrak{N}_0(N) & \longrightarrow & 0
\end{array}$$

(compare (I.7)).

Since $\omega(\alpha,\beta) = 0$ we may choose $d \in \text{incl}_2^{-1}(\omega(\widetilde{\alpha},\beta))$ and even a monomorphism $u\colon \alpha \hookrightarrow \beta$ such that $d = d(u,-u)$ (cf. Theorem D). But—because of our assumption on α —this difference invariant has a very special form: it lies in the image of the homomorphism

$$i_*\colon \Omega_2(N;\eta) \longrightarrow \Omega_2(P(\alpha);\phi)$$

induced by the inclusion $N = P(0 \times \underset{\sim}{\mathbb{R}}) \subset P(\alpha)$. Indeed, due to the complex structure of α' we can rotate $u \mid \alpha'$ in its image to $-u \mid \alpha'$. So it remains only to deform $u \mid \underset{\sim}{\mathbb{R}}$ in the complement $u(\alpha')^\perp$ (of $u(\alpha')$ in β) to $-u \mid \underset{\sim}{\mathbb{R}}$. As in Proposition 1.5 we see that

$$d = d(u,-u) = \pm i_*(\omega(\underset{\sim}{\mathbb{R}}, u\,(\alpha)^\perp))\,. \tag{5.12}$$

This suggests to consider also the exact singularity sequence (cf. [K 1], 9.3)

$$\mathbb{Z}_2 \xrightarrow{\delta\delta_1'} \Omega_2(N;\eta) \xrightarrow{\text{forg}_2 \circ i_*} \Omega_2(N) \tag{5.13}$$

which is compatible with (the generalization of) (5.4) and (5.4′) via i_*; the $H_1(P^\infty)$-term in (5.4′) plays no role here and hence the Θ_1-term takes the form $\mathbb{Z}_2 \cdot \delta_1'(1)$ and vanishes precisely if $w_2(\eta) \neq 0$.

Now recall that by assumption $0 = \overline{\omega}(\widetilde{\alpha}, \beta) = \overline{\mathrm{incl}}_2(\mathrm{forg}_2(d))$. Therefore we may choose $\overline{e} \in \Omega_2(N)$ such that $2\overline{e} = \mathrm{forg}_2(d)$. If $w_2(\eta)(\overline{e}) = 0$ and hence $\overline{e} = \mathrm{forg}_2(e)$ for some bordism class $e \in \Omega_2(P(\alpha); \phi)$, we may replace d and u by $\widehat{d}$ and $\widehat{u}$ in the whole previous discussion where

$$\widehat{d} = d(\widehat{u}, -\widehat{u}) := d - \partial_*(e).$$

But then

$$0 = \mathrm{forg}_2(\widehat{d}) = \pm\, \mathrm{forg}_2 \circ i_* \big(\omega(\underset{\sim}{\mathbb{R}}, \widehat{u}(\alpha)^\perp)\big)$$

and $\widehat{d}$ is a multiple of $\delta\delta_1'(1)$, which may survive in $\Omega_2(P(\alpha); \phi)$ but certainly doesn't in $\Omega_2(P(\widetilde{\alpha}); \widetilde{\phi})$ (because $w_2(\widetilde{\phi}) \neq 0$).

We conclude that $\omega(\widetilde{\alpha}, \beta) = \mathrm{incl}_*(\widehat{d})$ vanishes *provided* the element $\overline{e} \in \Omega_2(N)$ discussed above can be chosen so that $w_2(\eta)(\overline{e}) = 0$. This is certainly the case if $w_2(\eta) \equiv 0$ on all of $\Omega_2(N)$ or $w_2(\eta) \not\equiv 0$ on the elements of order 2 in $\Omega_2(N)$. Otherwise, diagram chasing in (5.11) shows that

(5.14) $$\widetilde{\omega} := w_2(\eta)(\overline{e}) \in \mathbb{Z}_2$$

is a well-defined extra antipodality obstruction (and actually the only one), independent of the choices of d and $\overline{e}$. For example, given any monomorphism $u\colon \alpha \hookrightarrow \beta$ (whose existence is guaranteed by Theorem A), put $d = d(u, -u)$ (so that by (5.11)–(5.13)

$$\pm\, \mathrm{forg}_2(d) = \mathrm{forg}_2(\omega(\underset{\sim}{\mathbb{R}}, u(\alpha)^\perp)) = \overline{\omega}(\underset{\sim}{\mathbb{R}}, u(\alpha)^\perp) \in 2 \cdot \Omega_2(N))$$

and obtain $\widetilde{\omega}$ by evaluating $w_2(\eta)$ on any element $\overline{e} \in \Omega_2(N)$ such that $2\overline{e} = \overline{\omega}(\underset{\sim}{\mathbb{R}}, u(\alpha)^\perp)$. Note that this last expression corresponds to the Euler class $e(\underset{\sim}{u}(\alpha)^\perp)$ via Hurewicz and Poincaré duality isomorphisms (cf., e.g., [K 1], 5.3).

Remark 5.15: In the proof above we have dealt with that part of the antipodality obstruction $\omega(\widetilde{\alpha}, \beta)$ which lies in $\ker \widetilde{\mathrm{forg}}_2$ and hence in the image of $H_1(S^1) \oplus H_1(P^\infty)$ (see the left hand term of the middle line in diagram (5.11); the summand $\mathbb{Z}_2$ plays no role in $\Omega_2(P(\widetilde{\alpha}); \widetilde{\phi})$ since $\widetilde{\overline{\tau}}$ is onto as in (5.6)). Our extra invariant $\widetilde{\omega}$ (cf. (5.14)) is well-defined precisely when the $H_1(S^1)$-part of $\omega(\widetilde{\alpha}, \beta)$ is defined without indeterminacy but does *not* survive in $\omega(\alpha, \beta)$ via $\pitchfork$, and then these two $\mathbb{Z}_2$-obstructions coincide. The $H_1(P^\infty)$-part of $\omega(\widetilde{\alpha}, \beta)$ vanishes due to the special form of α. ∎

It remains to make the condition $\omega(\alpha,\beta) = 0$ more accessible — at least in some interesting cases.

PROPOSITION 5.16: *Let $\alpha^3 = \alpha' \oplus \underset{\sim}{\mathbb{R}}$ and $\beta^{n+1} = \beta' \oplus \underset{\sim}{\mathbb{R}}$ be vector bundles over an n-manifold N^n of even dimension $n = 2q > 4$ (where α' and β' are complex vector bundles).*

In case $n \equiv 0(4), w_1(N) = 0$ and $w_2(\alpha) + w_2(\beta) + w_2(N) \equiv 0$ on $H_2(N;\mathbb{Z})$, we have: there exists a monomorphism from α to β if and only if

$$c_q(\beta' - \alpha')[N] \equiv 0(4).$$

In all other cases, such a monomorphism exists precisely if $w_n(\beta - \alpha) = 0$.

Proof: Given a generic complex vector bundle homomorphism $u'\colon \alpha' \to \beta'$ (or, equivalently, a generic section of the homomorphism bundle $\underset{\sim}{\mathrm{Hom}}_{\mathbb{C}}(\alpha',\beta')$), we may push all its zeroes into a small ball $B \subset N$ and then count them algebraically by

$$c_q (= c_q(\beta' - \alpha')[N] = c_q(\underset{\sim}{\mathrm{Hom}}_{\mathbb{C}}(\alpha',\beta'))[N]) \in \mathbb{Z} = \pi_{n-1}(S^{n-1})$$

(at least if N is orientable). The corresponding homomorphism $u\colon \alpha' \oplus \underset{\sim}{\mathbb{R}} \to \beta' \oplus \underset{\sim}{\mathbb{R}}$ has local obstruction $j_*(c_q)$ where the composite map

$$j\colon S^{n-1} \longrightarrow V_{n,2} \subset V_{n+1,3}$$

between Stiefel manifolds involves the complex structure on $\mathbb{R}^n = \mathbb{C}^q$.

According to Paechter's tables (cf. [P], p. 249) $\pi_{n-1}(V_{n+1,3})$ is isomorphic to $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $n \equiv 0(4)$ and $n \equiv 2(4)$, respectively. Under the inclusion $i\colon B \subset N$ this corresponds to the group built up from the $\mathbb{Z}_2$-terms in the singularity sequence

$$\overline{\Omega}_2(N;\eta) \oplus \mathfrak{N}_1(N) \xrightarrow{\sigma j_2} \mathbb{Z}_2 \longrightarrow \Omega_1(P(\alpha);\phi) \longrightarrow \overline{\Omega}_1(N;\eta) \oplus \mathbb{Z}_2 \to 0.$$

If $\sigma j_2 \not\equiv 0$ or $n \equiv 2(4)$, then the global obstruction $\omega(\alpha,\beta) = i_* j_*(c_q)$ contains precisely as much information as its image in the $\mathbb{Z}_2$-term to the right, i.e., $w_n(\beta - \alpha)[N]$. However, if $n \equiv 0(4)$ and $\sigma j_2 \equiv 0$ (i.e., $w_1(N) = 0$ and $w_2(\eta) \equiv 0$ on $H_2(N;\mathbb{Z})$), then

$$i_*\colon \pi_{n-1}(V_{n+1,3}) = \mathbb{Z}_4 \hookrightarrow \Omega_1(P(\alpha);\phi)$$

maps the mod 4 class of c_q injectively to $\omega(\alpha,\beta)$. ∎

Finally, let us apply the results of this section to the situation in Example G of the introduction: $\alpha = \alpha' \oplus \underset{\sim}{\mathbb{R}}$ and $\beta = \beta' \oplus \underset{\sim}{\mathbb{R}}$ where α' and β' are the complex bundles λ_p and $\oplus\lambda_{p_i}$ over $N = \mathbb{C}P(q), q > 2$. As in (5.3) put $z = c_1(\lambda_1)$ and $z_2 = w_2(\lambda_1)$. Then we have

$$w_2(\eta) = (p + \Sigma p_i + q + 1)z_2 \quad \text{and}$$
$$c_q(\beta' - \alpha') = c_q(\overline{\alpha}' \otimes_{\mathbb{C}} \beta') = \prod (p_i - p) z^q.$$

First assume that $p_i \equiv p(2)$ for some i, say $i = 1$, or, equivalently, that

$$w_n(\beta - \alpha) = \prod (p_i - p) z_2^q$$

vanishes. Then so does $\omega(\alpha, \beta)$ except when $p_1 - p \equiv 2(4), p_2 \equiv \cdots \equiv p_q \not\equiv 0(2)$ and $p \equiv q \equiv 0(2)$ (see Proposition 5.16). But this exceptional case gets excluded anyway if we require the $(n-2)$-dimensional part of

$$\begin{aligned} w(\beta - \alpha) =& (1 + pz_2)^{-1} \cdot \prod (1 + p_i z_2) \\ =& (1 + p_2 z_2) \dots (1 + p_q z_2) \end{aligned}$$

to be trivial. Thus $\omega(\alpha, \beta)$ and $\overline{\omega}(\widetilde{\alpha}, \beta)$ — or, equivalently, $w_n(\beta - \alpha)$ and $w_{n-2}(\beta - \alpha)$ — vanish precisely if $p_i \equiv p(2)$ and $p_j \equiv 0(2)$ for some $i \neq j$. By Theorem 5.10 this necessary condition is also fully sufficient for the existence of an antipodal monomorphism except when $w_2(\eta) \neq 0$, and hence the parity of $\overline{e} = \frac{1}{2} PD\left(e(u(\alpha)^{\perp})\right)$ defines the extra obstruction $\widetilde{\omega}$ (cf. (5.14)). For example, if $p_1 = p$ and $p_2 \equiv 0(2)$ but $w_2(\eta) \neq 0$, then we may choose u so that $u(\alpha)^{\perp} = \lambda_{p_2} \oplus \dots \oplus \lambda_{p_q}$; thus in this special case $\widetilde{\omega} = 0$ precisely if

$$c_{q-1}(u(\alpha)^{\perp}) = p_2 \dots p_q z^{q-1}$$

is divisible by 4.

The classification statement in Example G follows from Proposition 5.8. ∎

§6. Nonstable and stable tangent plane fields on complex projective spaces

In this section we will prove the statements which were listed in Example H of the introduction.

Let p and q be integers, $q \geq 1$. As before, λ_p will denote the p-fold (complex) tensor power of the canonical complex line bundle over $N^n = \mathbb{C}P(q)$, $z := c_1(\lambda_1)$ and $z_2 := w_2(\lambda_1)$.

A complex monomorphism from λ_p to TN exists precisely if the homomorphism bundle

$$\underset{\sim}{\mathrm{Hom}}_{\mathbb{C}}(\lambda_p, TN) \cong \lambda_{-p} \otimes (\lambda_{-1} \otimes \lambda_1^{\perp})$$

(cf. (5.2)) has a nowhere vanishing section or, equivalently, a trivial Euler or top Chern class c_q. The total Chern class equals

$$\begin{aligned} c(\underset{\sim}{\mathrm{Hom}}(\lambda_p, TN)) =& c(\lambda_{-p} \otimes \lambda_{-1} \otimes \underset{\sim}{\mathbb{C}}^{q+1} - \lambda_{-p}) \\ =& ((1-pz) - z)^{q+1}(1-pz)^{-1} \\ =& \sum_{i=0}^{q} \binom{q+1}{i+1} (1-pz)^i (-z)^{q-i}. \end{aligned}$$

Its top component

$$c_q = \sum_{i=0}^{q} \binom{q+1}{i+1} p^i \cdot (-z)^q \tag{6.1}$$

vanishes precisely if $p \neq 0$ and

$$p \cdot \sum_{i=0}^{q} \binom{q+1}{i+1} p^i = (1+p)^{q+1} - 1 = 0,$$

i.e., if $p = -2$ and q odd. This proves the existence claim in Example H, (i), whenever $q \geq 1$.

A similar but simpler mod 2 calculation shows that

$$w_n(TN - \lambda_p) = 0 \tag{6.2}$$

if and only if $p \equiv 0(2)$ and $q \equiv 1(2)$. This necessary condition for the existence of a real monomorphism $\lambda_p \hookrightarrow TN$ (nonstable or not) implies also that

$$w_2(\phi_{\mathbb{C}}) = w_2(\lambda_{-p} \otimes_{\mathbb{C}} TN - TN) = pqz_2 = 0$$

and

$$w_2(\phi) = (q-1)x^2 + pz_2 = 0$$

(cf. (5.1)). Hence for $q \geq 2$ it follows from [K 4], 4.3, that a (nonstable) real monomorphism from λ_p to TN exists precisely if p is even and

$$\pm c_q(\underset{\sim}{\mathrm{Hom}}(\lambda_p, TN))[N] \equiv q + 1 + \binom{q+1}{2} p \equiv 0(2p)$$

(compare (6.1)); for $q = 4r - 1$ (and $q = 4r + 1$, resp.) this just means that

$$q + 1 \equiv -\binom{q+1}{2} p = -2r(4r-1)p \equiv 0 \ (2p)$$

(and $q + 1 \equiv -(2r+1)(4r+1)p \ (2p)$, resp.). This proves the existence claim in Example H, (ii).

In view of the observation (6.2) the remaining existence claims in (iii) and (iv) now follow from Proposition 5.16 and Theorem 5.10 and from the fact that for even p

$$v_{n-2}(\lambda_p \oplus \underset{\sim}{\mathbb{R}}, T\mathbb{C}P(q) \oplus \underset{\sim}{\mathbb{R}}) = w_{n-2}(\mathbb{C}P(q)) = \binom{q+1}{2} z_2^{q-1}.$$

Finally, recall (from [K 4], 1.5 and 4.6) that complex monomorphisms $\lambda_p \hookrightarrow TN$ —if they exist and if $q \geq 2$— correspond to the elements of $\Omega_1(N; \phi_{\mathbb{C}}) \cong \mathbb{Z}_2$; moreover, the transition homomorphism ϱ_2 to $\Omega_2(P(\alpha); \phi)$ is injective and has cokernel $\mathbb{Z}_2$. This completes the proof of the statements (i) and (ii) in Example H. In addition, we conclude that if two complex monomorphisms are regularly homotopic in the real sense then they allow also a complex regular homotopy.

The remaining enumeration statements in Example H follow from Theorem D, (5.4), Proposition 5.8 and (1.4). ∎

Example 6.3: Given $q \geq 1$, q odd, a real monomorphism from λ_p to $T\mathbb{C}P(q)$ exists at least if $p = \pm 2$. Here is a complete list of all other such p for a few low values of q.

q	1 or 3	5 or 11	7	9 or 19	13 or 27	15	17 or 35
p	—	± 6	± 4	± 10	± 14	± 4 or ± 8	± 6 or ± 18

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